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NOTE

Determinacy of Bounded Complex Perturbations of Jacobi Matrices

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We consider complex Jacobi matrices G which can be decomposed in the form $G=J+C$, where J is a real Jacobi matrix and C is a complex Jacobi matrix whose entries are uniformly bounded. We prove that the determinacy of the operator defined by G is equivalent to that of J. From this we deduce that the determinacy of G is equivalent to the coincidence between the domains of definition of the operators G and its adjoint G^*.

1. INTRODUCTION

In this paper, we study operators given by a tridiagonal matrix

$$G=\begin{pmatrix} b_0 & a_1 & 0 & \cdots \\ a_1 & b_1 & a_2 & \cdots \\ 0 & a_2 & b_2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{1}$$

with $a_n \in \mathbb{C}\backslash\{0\}$, and $b_n \in \mathbb{C}$, which is called a (complex) Jacobi matrix. During the last few years the properties of operators connected with infinite matrices with complex entries have attracted some attention (see [2–7]). This is due in part to their application to the study of the convergence of continued fractions, Padé, and Hermite-Padé approximations.

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In [9, p. 99] the notion of determinacy of a Jacobi matrix G with complex entries was introduced. By analogy with the real case (see [1, p. 19]), the matrix (1) is determinate if there exists a $z \in \mathbb{C}$ such that

$$\sum_{n \geqslant 1} |p_n(z)|^2 = \infty \qquad \text{or} \qquad \sum_{n \geqslant 1} |q_n(z)|^2 = \infty, \tag{2}$$

where $p_n(z)$ and $q_n(z)$ are the solutions of the recurrence relations

$$a_{n+1}\, y_{n+1} + b_n\, y_n + a_n\, y_{n-1} = z y_n, \qquad n = 0, 1, 2, \ldots \qquad (a_0 = 1), \tag{3}$$

with the initial conditions

$$\begin{cases} p_{-1}(z) = 0, & p_0(z) = 1, \\ q_0(z) = 0, & q_1(z) = \dfrac{1}{a_1}. \end{cases} \tag{4}$$

A sufficient condition for the determinate case to hold (see [5, Lemma 1]) is that either

$$\sum_{n \geqslant 0} \frac{1}{|a_n|} = \infty \qquad \text{or} \qquad \sum_{n \geqslant 0} \frac{|b_n|}{|a_n a_{n+1}|} = \infty. \tag{5}$$

The concept of determinacy was used in [5] to prove the convergence of the continued fraction (Padé approximants) associated with (1). One of the aims of this paper is to clarify the "operator" meaning of the determinacy condition (2) for complex Jacobi matrices.

The infinite matrix G defines, by the usual operation of a matrix on a vector, an operator on the linear subspace of ℓ^2 formed by all vectors which have a finite number of components different from zero. Let G also denote the closure of this operator in ℓ^2 and G^* its adjoint. By $D(\cdot)$, we denote the domain of definition of the operator $(\cdot)$. In the case when $a_n \in \mathbb{R} \setminus \{0\}$ and $b_n \in \mathbb{R}$ (that is, for real Jacobi matrices), it is known (see [1, pp. 138–141; 8, p. 76]) that the determinate case holds if and only if

$$D(G) = D(G^*). \tag{6}$$

Taking into consideration the property of symmetry of a real Jacobi matrix, (6) means that the closed operator G is selfadjoint.

Here, we prove that for a large class of complex Jacobi matrices, determinacy and property (6) remain equivalent. More precisely, we have

THEOREM 1. *If the complex Jacobi matrix* (1) *admits the decomposition*

$$G = J + C, \tag{7}$$

where J is a real Jacobi matrix and C has uniformly bounded complex entries, then (2) *and* (6) *are equivalent.*

Theorem 1 is a direct consequence of Theorem 2.

THEOREM 2. *If the complex Jacobi matrix* (1) *admits the decomposition* (7) *where J is a real Jacobi matrix and C has uniformly bounded complex entries, then the determinacy of the matrices G and J are equivalent.*

The fact that the determinacy of J implies that of G was proved in Lemma 3 of [5]. Some sufficient conditions for the reversed statement were also given (see Lemma 4 and Remark 1 of [5]), whereas the proof that it is true in general was posed as an open problem. We present a proof of this fact in the next section. Here we deduce Theorem 1 from Theorem 2.

Proof of Theorem 1. Since C defines a bounded operator on all ℓ^2, we have that $D(C)=D(C^*)=\ell^2$. Because of (7), it follows that $D(G)=D(J)$ and $D(G^*)=D(J^*)$. Hence, $D(G)=D(G^*)$ if and only if $D(J)=D(J^*)$. But, as was mentioned above, in the case of real Jacobi matrices, this is equivalent to the determinacy of J which in turn by Theorem 2 is equivalent to the determinacy of G. ∎

2. PROOF OF THEOREM 2

Before proceeding with the proof of Theorem 2, let us make some reductions and introduce new notation. A direct consequence of the Theorem of Invariability (see [9, p. 96]), is that a bounded perturbation on the coefficients b_n does not alter the condition of determinacy of a Jacobi matrix. Since the coefficients of C (and the imaginary parts of the coefficients of G) are bounded, there is no loss of generality if we restrict our attention to the case when the coefficients b_n of G are real numbers and they coincide with the corresponding coefficients of J. Thus, the coefficients of G are $a_n \in \mathbb{C}\setminus\{0\}$ and $b_n \in \mathbb{R}$, while the corresponding ones of J are $\tilde{a}_n \in \mathbb{R}\setminus\{0\}$ and b_n.

By $p_n^{(k)}(z)$, $k=0, 1, ...,$ we denote the normalized associated polynomial of type k and degree n relative to (3). Such polynomials are defined as the solution of

$$\begin{aligned} a_{n+k+1}p_{n+1}^{(k)}(z) &= (z-b_{n+k})\, p_n^{(k)}(z) - a_{n+k}p_{n-1}^{(k)}(z), \qquad n \geqslant 0, \\ p_{-1}^{(k)}(z) &= 0, \qquad p_0^{(k)}(z) = 1/a_k. \end{aligned} \tag{8}$$

Notice that $p_n^{(0)}(z)=p_n(z)$ and $p_{n-1}^{(1)}(z)=q_n(z)$, since we took $a_0=1$. Analogously, we denote by $\tilde{p}_n^{(k)}(z)$ the associated polynomial of type k and degree n which is the solution of (8) when we replace a_n by $\tilde{a}_n$. These polynomials correspond to J. Finally, let

$$P_n^{(k)}(z) = a_k \cdots a_{n+k} p_n^{(k)}(z) \qquad \text{and} \qquad \tilde{P}_n^{(k)}(z) = \tilde{a}_k \cdots \tilde{a}_{n+k} \tilde{p}_n^{(k)}(z)$$

be the corresponding monic polynomials. For future reference, we would like to underline that the polynomials $\tilde{P}_n^{(k)}(z)$ satisfy the recurrence relations

$$\begin{aligned} &\tilde{P}_{n+1}^{(k)}(z) = (z - b_{n+k})\, \tilde{P}_n^{(k)}(z) - \tilde{a}_{n+k}^2\, \tilde{P}_{n-1}^{(k)}(z), \qquad n \geqslant 0, \\ &\tilde{P}_{-1}^{(k)}(z) = 0, \qquad \tilde{P}_0^{(k)}(z) = 1. \end{aligned} \tag{9}$$

Let us assume that J is indeterminate. Because of the Theorem of Invariability this is equivalent to

$$\sum_{n \geqslant 1} |\tilde{p}_n(0)|^2 < +\infty \qquad \text{and} \qquad \sum_{n \geqslant 1} |\tilde{p}_n^{(1)}(0)|^2 < +\infty. \tag{10}$$

We must prove that relations (10) are also true at $z = 0$ for the polynomials $p_n(z)$ and $p_n^{(1)}(z)$. We will prove this using a relation between the polynomials $\tilde{p}_n(z)$, $\tilde{p}_n^{(1)}(z)$ and $p_n(z)$, $p_n^{(1)}(z)$. In order to simplify the notation, in the sequel we denote

$$p_n^{(k)}(0) = p_n^{(k)}, \qquad P_n^{(k)}(0) = P_n^{(k)}, \qquad \tilde{p}_n^{(k)}(0) = \tilde{p}_n^{(k)}, \qquad \tilde{P}_n^{(k)}(0) = \tilde{P}_n^{(k)}.$$

The following relation is the key in the proof. For all $n \geqslant 2$, we have

$$p_n^{(m)} = \frac{\tilde{P}_n^{(m)}}{a_m \cdots a_{m+n}} + \sum_{k=2}^{n} \frac{\tilde{a}_{m+k-1}^2 - a_{m+k-1}^2}{a_{m+k-1}} \frac{\tilde{P}_{n-k}^{(m+k)}}{a_{m+k} \cdots a_{m+n}} p_{k-2}^{(m)}. \tag{11}$$

Fix m, the equality can be proved by induction on the parameter n. Using (8) it is easy to verify that (11) holds for $n = 2$ and $n = 3$. Let us assume that (11) is true for all values of the parameter up to $n - 1$, $n \geqslant 4$, we will show that it also holds for n.

Using (8) with $z = 0$, we have

$$\begin{aligned} p_n^{(m)} = &-\frac{b_{m+n-1}}{a_{m+n}} p_{n-1}^{(m)} - \frac{\tilde{a}_{m+n-1}^2}{a_{m+n-1} a_{m+n}} p_{n-2}^{(m)} \\ &+ \frac{\tilde{a}_{m+n-1}^2 - a_{m+n-1}^2}{a_{m+n-1}} \frac{1}{a_{m+n}} p_{n-2}^{(m)}. \end{aligned} \tag{12}$$

Substituting in the first two terms to the right of (12) the expressions given by (11) for $n - 2$ and $n - 1$ in place of n, and rearranging the sums conveniently, we obtain

$$p_n^{(m)} = \frac{-b_{m+n-1}\tilde{P}^{(m)}_{n-1} - \tilde{a}^2_{m+n-1}\tilde{P}^{(m)}_{n-2}}{a_m \cdots a_{m+n}}$$

$$+\sum_{k=2}^{n-2} \frac{\tilde{a}^2_{m+k-1} - a^2_{m+k-1}}{a_{m+k-1}} \frac{-b_{m+n-1}\tilde{P}^{(m+k)}_{n-k-1} - \tilde{a}^2_{m+n-1}\tilde{P}^{(m+k)}_{n-k-2}}{a_{m+k} \cdots a_{m+n}} p^{(m)}_{k-2}$$

$$+\frac{\tilde{a}^2_{m+n-2} - a^2_{m+n-2}}{a_{m+n-2}} \frac{-b_{m+n-1}\tilde{P}^{(m+n-1)}_0}{a_{m+n-1}a_{m+n}} p^{(m)}_{n-3}$$

$$+\frac{\tilde{a}^2_{m+n-1} - a^2_{m+n-1}}{a_{m+n-1}} \frac{1}{a_{m+n}} p^{(m)}_{n-2}.$$

On account of (9), this formula reduces to (11).

Proof of Theorem 2. If the first condition in (5) takes place, it is easy to conclude that both G and J are determinate (Lemmas 1 and 4 of [5]). Therefore, we may assume that

$$\sum_{n \geqslant 0} \frac{1}{|a_n|} < \infty. \tag{13}$$

In particular, $\lim_{n\to\infty} |a_n| = \infty$. Under this condition, from the boundedness of the coefficients in C, it is easy to check that there exists a constant $M_1 > 0$ such that

$$\left|\frac{\tilde{a}^2_k - a^2_k}{a_k}\right| = \frac{|\tilde{a}_k - a_k|\,|\tilde{a}_k + a_k|}{|a_k|} \leqslant M_1 < \infty, \qquad k = 0, 1, \ldots . \tag{14}$$

On the other hand, from (13) and the boundedness of the coefficients in C, it follows that for any $n \in 0, 1, \ldots$ and $k = 0, \ldots, n$

$$\left|\frac{\tilde{a}_k \cdots \tilde{a}_n}{a_k \cdots a_n}\right| \leqslant \prod_{j=k}^{n} \left(1 + \left|\frac{\tilde{a}_j - a_j}{a_j}\right|\right) \leqslant \lim_{n\to\infty} \prod_{j=0}^{n} \left(1 + \left|\frac{\tilde{a}_j - a_j}{a_j}\right|\right) \leqslant M_2 < \infty. \tag{15}$$

Let $m \in 0, 1, \ldots, n = 2, 3, \ldots$ and $k = 2, \ldots, n$. Set

$$\alpha^{(m)}_{n,k} = \frac{\tilde{a}^2_{m+k-1} - a^2_{m+k-1}}{a_{m+k-1}} \frac{\tilde{P}^{(m+k)}_{n-k}}{a_{m+k} \cdots a_{m+n}}, \qquad \beta^{(m)}_n = \frac{\tilde{P}^{(m)}_n}{a_m \cdots a_{m+n}}.$$

With this notation, we can write (11) as

$$p^{(m)}_n = \beta^{(m)}_n + \sum_{k=2}^{n} \alpha^{(m)}_{n,k}\, p^{(m)}_{k-2}, \qquad n \geqslant 2. \tag{16}$$

We are particularly interested in this formula for $m=0, 1$. On account of (16), it is known (see [1, Lemma 1.3.2]) that from

$$\sum_{n\geq 2} |\beta_n^{(m)}|^2 < +\infty \qquad \text{and} \qquad \sum_{n\geq 2} \sum_{k=2}^{n} |\alpha_{n,k}^{(m)}|^2 < +\infty \tag{17}$$

it follows that $\sum_{n\geq 1} |p_n^{(m)}|^2 < +\infty$. Thus in order to conclude the proof it remains to verify that (17) holds for $m=0, 1$.

From (15), it follows that

$$|\beta_n^{(m)}| = \left|\frac{\tilde{a}_m \cdots \tilde{a}_{m+n}}{a_m \cdots a_{m+n}}\right| |\tilde{p}_n^{(m)}| \leqslant M_2 \, |\tilde{p}_n^{(m)}|.$$

From (10), it follows that $\sum_{n\geq 2} |\beta_n^{(m)}|^2 < +\infty$, $m=0, 1$. On the other hand, from (14) and (15), we have

$$|\alpha_{n,k}^{(m)}| = \left|\frac{\tilde{a}_{m+k-1}^2 - a_{m+k-1}^2}{a_{m+k-1}}\right| \left|\frac{\tilde{a}_{m+k} \cdots \tilde{a}_{m+n}}{a_{m+k} \cdots a_{m+n}}\right| |\tilde{p}_{n-k}^{(m+k)}| \leqslant M_1 M_2 \, |\tilde{p}_{n-k}^{(m+k)}|.$$

Therefore, making use of the well known formula (see, for example, formula (17) in [5])

$$(\tilde{p}_{m+n-1}^{(1)} \tilde{p}_{m+k-1}^{(0)} - \tilde{p}_{m+n}^{(0)} \tilde{p}_{m+k-2}^{(1)}(z)) = \tilde{p}_{n-k}^{(m+k)}(z),$$

and (10), it follows that

$$\begin{aligned} \sum_{n\geq 2} \sum_{k=2}^{n} |\alpha_{n,k}^{(m)}|^2 &\leqslant M_1 M_2 \sum_{n\geq 2} \sum_{k=2}^{n} |\tilde{p}_{m+n-1}^{(1)} \tilde{p}_{m+k-1}^{(0)} - \tilde{p}_{m+n}^{(0)} \tilde{p}_{m+k-2}^{(1)}|^2 \\ &\leqslant 2M_1 M_2 \left(\sum_{n\geq 2} |\tilde{p}_{m+n-1}^{(1)}|^2 \sum_{k=2}^{n} |\tilde{p}_{m+k-1}^{(0)}|^2 \right. \\ &\quad \left. + \sum_{n\geq 2} |\tilde{p}_{m+n}^{(0)}|^2 \sum_{k=2}^{n} |\tilde{p}_{m+k-2}^{(1)}|^2 \right) \\ &\leqslant 4M_1 M_2 \sum_{n\geq 0} |\tilde{p}_n^{(0)}|^2 \sum_{k\geq 0} |\tilde{p}_k^{(1)}|^2 < +\infty. \end{aligned}$$

The theorem is proved. ∎

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